

Example 1

The age-independent, five-year survival probability for a Scottish male diagnosed with pancreatic cancer is 3.5% (Source: ISD Scotland press release, August 2004).

(i) What constant force of mortality is implied by this?

(ii) What is the one-year survival probability?

(iii) What distribution of future lifetime is implied by a constant force of mortality?

Solution

(i) Using the basic general result linking the survival probability to the force of mortality:

$$\begin{aligned} S(t) &= \exp\left(-\int_0^t \mu_s ds\right) \\ &= \exp\left(-\int_0^t \mu ds\right) \\ &= \exp(-t\mu) \end{aligned}$$

So $S(5) = e^{-5\mu}$.

We are given that $S(5) = 0.035$.

Rearranging gives constant force of mortality, $\mu = -\frac{1}{5} \log 0.035 = 0.670$.

(ii)

$$\begin{aligned} S(1) &= e^{-1 \cdot \mu} \\ &= e^{-0.670} \\ &= 0.511 \end{aligned}$$

So, one-year survival probability is 51.1%.

(iii)

$$\begin{aligned} F(t) &= 1 - S(t) \\ &= 1 - e^{-t\mu} \end{aligned}$$

which we recognise as the distribution function of the exponential distribution.

Example 2

The probability of a British male non-smoking doctor born in 1900–1909 surviving from age 35 to age 70 was 76%. The same probability for an equivalent smoker was 58% (Source: Doll et al, British Medical Journal, 2004).

(i) What is the constant additional force of mortality for smokers implied by this over the age range 35–70?

(ii) The survival probabilities to age 90 were 18% for non-smokers and 4% for smokers. What is the constant additional force of mortality for smokers implied by this over the age range 35–90?

(iii) What is implied about our assumption of a constant additional force of mortality?

Solution

(i) Recall the general result:

$${}_t p_x = \exp\left(-\int_0^t \mu_{x+s} ds\right)$$

and assume that $\mu_{x+s}^{\text{smoker}} = \mu_{x+s}^{\text{nonsmoker}} + \lambda$.

$$\begin{aligned} {}_t p_x^{\text{smoker}} &= \exp\left(-\int_0^t [\mu_{x+s}^{\text{nonsmoker}} + \lambda] ds\right) \\ &= \exp\left(-\int_0^t \mu_{x+s}^{\text{nonsmoker}} ds\right) \cdot \exp\left(-\int_0^t \lambda ds\right) \\ &= {}_t p_x^{\text{nonsmoker}} \cdot e^{-t\lambda} \\ \Rightarrow e^{-t\lambda} &= \frac{{}_t p_x^{\text{smoker}}}{{}_t p_x^{\text{nonsmoker}}} \\ \Rightarrow \lambda &= -\frac{1}{t} \log\left(\frac{{}_t p_x^{\text{smoker}}}{{}_t p_x^{\text{nonsmoker}}}\right) \end{aligned}$$

We are given that ${}_{35} p_{35}^{\text{nonsmoker}} = 0.76$ and ${}_{35} p_{35}^{\text{smoker}} = 0.58$.

$$\text{So } \lambda = -\frac{1}{35} \log\left(\frac{0.58}{0.76}\right) = 0.00772$$

$$\text{(ii) } \lambda = -\frac{1}{55} \log\left(\frac{0.04}{0.18}\right) = 0.0273$$

(iii) The additional force of mortality due to smoking does not look to be constant.

$${}_t p_x = \exp\left(-\int_0^t \mu_{x+s} ds\right)$$

Proof 1

$$\begin{aligned} {}_{t+dt} p_x &= {}_t p_x \cdot dt p_{x+t} \\ &= {}_t p_x (1 - dt q_{x+t}) \\ &= {}_t p_x (1 - dt \cdot \mu_{x+t} + o(dt)) \end{aligned} \tag{1}$$

Remember $\lim_{dt \rightarrow 0^+} \frac{o(dt)}{dt} = 0$.

Rearranging (1) gives:

$${}_{t+dt} p_x - {}_t p_x = -{}_t p_x \cdot dt \cdot \mu_{x+t} + o(dt)$$

and so:

$$\frac{{}_{t+dt} p_x - {}_t p_x}{dt} = -{}_t p_x \mu_{x+t} + \frac{o(dt)}{dt}$$

Taking the limit as $dt \rightarrow 0$:

$$\frac{\partial}{\partial t} {}_t p_x = -{}_t p_x \mu_{x+t} \tag{2}$$

This is a differential equation. Remember the general result:

$$\frac{\partial}{\partial t} e^{G(t)} = e^{G(t)} \cdot \frac{\partial}{\partial t} G(t)$$

and recall the result from basic calculus that:

$$\frac{\partial}{\partial t} \int_0^t g(s) ds = g(t)$$

Thus:

$$\frac{\partial}{\partial t} \int_0^t \mu_{x+s} ds = \mu_{x+t}$$

and therefore the solution to (2) is:

$${}_t p_x = \exp\left(-\int_0^t \mu_{x+s} ds\right)$$

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Proof 2

$$\begin{aligned} {}_t p_x &= \prod_{i=0}^{t-1} {}_1 p_{x+i} \\ &= \prod_{j=0}^{\frac{t}{h}-1} {}_h p_{x+j} \end{aligned}$$

for some small h . Now ${}_h p_{x+j} = 1 - {}_h q_{x+j}$ and ${}_h q_{x+j} \approx h \cdot \mu_{x+j}$ for small h , so:

$$\begin{aligned} {}_t p_x &= \prod_{j=0}^{\frac{t}{h}-1} {}_h p_{x+j} \\ &= \prod_{j=0}^{\frac{t}{h}-1} (1 - {}_h q_{x+j}) \\ &= \prod_{j=0}^{\frac{t}{h}-1} (1 - h \cdot \mu_{x+j}) + o(h) \end{aligned}$$

Take natural logarithms of both sides, and note that $\log(1 - z) = -z + o(z)$ for small z :

$$\begin{aligned} \log {}_t p_x &= \sum_{j=0}^{\frac{t}{h}-1} \log(1 - h \cdot \mu_{x+j}) + o(h) \\ &= \sum_{j=0}^{\frac{t}{h}-1} -h \cdot \mu_{x+j} + o(h) \end{aligned}$$

Take the limit as $h \rightarrow 0^+$:

$$\log {}_t p_x = \int_0^t -\mu_{x+s} ds$$

which, after taking the exponent of each side, gives:

$${}_t p_x = \exp\left(-\int_0^t \mu_{x+s} ds\right)$$