2-state Markov model: fundamental assumptions

$$\text{Alive} \xrightarrow{\mu_x} \text{Dead}$$

1. **Markov assumption**. The probability that a life aged x will be in either state at any future time t depends only on the age x and the state currently occupied.
2. The probability ${}_{dt}q_{x+t} = dt\mu_{x+t} + o(dt), \quad dt > 0$

- First assumption means that the future depends only on the present, not on the past history.
- Second assumption is implied by the random-variable approach to future lifetime.

Multi-state Markov model

- Transition probability, ${}_tp_x^{gh} = \Pr(\text{In state } h \text{ at time } x+t | \text{In state } g \text{ at time } x)$
- Transition intensity μ_x^{gh}
- Probability never left state g is ${}_tp_x^{\overline{gg}}$
- *Not* the same thing as ${}_tp_x^{gg}$, which covers leaving state g and returning within t
- ${}_tp_x^{gg} = {}_tp_x^{\overline{gg}}$ only when return to state g is impossible

1. **Markov assumption**. The probability that a life aged x will be in a particular state at any future time t depends only on the age x and the state currently occupied.
2. The probability ${}_{dt}q_{x+t}^{gh} = dt\mu_{x+t}^{gh} + o(dt), \quad dt > 0$
3. The probability that a life makes two or more transitions in time dt is $o(dt)$.

Transition probabilities for Example 3.2 on page 19

- General three-state model would have following transition probabilities:

$$\begin{array}{cccc} & w & r & d \\ w & . & . & . \\ r & . & . & . \\ d & . & . & . \end{array}$$

- Which probabilities are zero by definition (see diagram on page 19)?

⇒No return to the working state, so ${}_tp_x^{rw} = {}_tp_x^{dw} = 0$
⇒No movement between the retired and dead states, so ${}_tp_x^{rd} = {}_tp_x^{dr} = 0$

$$\begin{array}{cccc} & w & r & d \\ w & . & . & . \\ r & 0 & . & 0 \\ d & 0 & 0 & . \end{array}$$

- Which probabilities are one by definition (see diagram)?

⇒No exit from retired or dead states, so so ${}_tp_x^{rr} = {}_tp_x^{dd} = 1$

(states from which there is no exit are called *absorbing states*)

$$\begin{array}{cccc} & w & r & d \\ w & . & . & . \\ r & 0 & 1 & 0 \\ d & 0 & 0 & 1 \end{array}$$

- Remaining transition probabilities to be found are thus:

$$\begin{array}{cccc} & w & r & d \\ w & {}_tp_x^{ww} & {}_tp_x^{wr} & {}_tp_x^{wd} \\ r & 0 & 1 & 0 \\ d & 0 & 0 & 1 \end{array}$$

- This is called the *transition matrix*
- Rows must sum to one, i.e. ${}_tp_x^{ww} + {}_tp_x^{wr} + {}_tp_x^{wd} = 1$

Kolmogorov forward equations

- Become familiar with deriving them — ideal examination material!
- For the transition probability, ${}_{t+dt}p_x^{gh}$, condition on all possible paths from state g to state h
- General formulæ:

$$\frac{\partial}{\partial t}{}_tp_x^{gh} = \sum_{j \neq h} {}_tp_x^{gj} \mu_{x+t}^{jh} - {}_tp_x^{gh} \sum_{j \neq h} \mu_{x+t}^{hj}, \quad g \neq h$$

and...

$${}_tp_x^{\overline{gg}} = \exp\left(- \int_0^t \sum_{j \neq g} \mu_{x+s}^{gj} ds \right)$$

where the latter is the generalisation of the two-state result:

$${}_tp_x = \exp\left(- \int_0^t \mu_{x+s} ds \right)$$

Data and estimation for Markov models

Notation

$$x_i = \text{exact age at which observation begins}$$
$$v_i = \text{time observed (called the } \textit{waiting time}\text{)}$$
$$d_i = \begin{cases} 0, & \text{if life is censored at end of waiting time} \\ 1, & \text{if life died at end of waiting time} \end{cases}$$

Example

A life born 14.3.1940 buys an annuity at 15.7.2005. At the end of 2005 the life office is carrying out an investigation of annuitant mortality and finds the annuitant still alive.

$x = 65.34$ years, being the age at observation start,
$v = 0.46$ years, being the time observed between 15.7.2005 and 31.12.2005,
$d = 0$, as life had not died by end of the waiting time.

At the time of the next investigation at the end of 2006 the life was still alive. This time the life's contribution to the mortality study is:

$x = 65.34$ years (the entry age is unchanged),
$v = 1.46$ years, being the time between 15.7.2005 and 31.12.2006, and
$d = 0$, as the life is still censored at the end of the waiting time.

At the time of the next investigation at the end of 2007 it was discovered that the life had died on 12.11.2007. This time the life's contribution to the mortality study is:

$x = 65.34$ years (the entry age is unchanged),
$v = 2.33$ years, being the time between 15.7.2005 and 12.11.2007, and
$d = 1$, as the life is dead at the end of the waiting time.

Inference

- We can think of d_i and v_i as realisations of random variables D_i and V_i
- V_i has a mixed distribution:
  (i) a continuous density where $d_i = 1$, and
  (ii) a mass of probability where $d_i = 0$.
- D_i and V_i are therefore **not independent**
- Assume $\mu_x = \mu, \forall x$, i.e. constant force of mortality
- The joint probability function, f_i, of D_i and V_i is $f_i(d_i, v_i) = e^{-\mu v_i}\mu^{d_i}$
- Of which basic probability function is this vaguely reminiscent?

$\Rightarrow$ Poission distribution with parameter μv_i, $f_i(d_i) = e^{-\mu v_i}.\dfrac{(\mu v_i)^{d_i}}{d_i!}$