Lecture cancelled on Friday 3rd February 2006

Data and estimation in the three-state model

• Consider the three-state model for working, death and retirement on page 34 and define the following random variables:

$$V_i = \text{waiting time for life } i$$
$$D_i = \begin{cases} 1 \text{ for dead} \\ 0 \text{ for not dead} \end{cases}$$
$$R_i = \begin{cases} 1 \text{ for retired} \\ 0 \text{ for not retired} \end{cases}$$

• Some observations:

(1) If $D_i = 1$, then $R_i = 0$

(2) If $R_i = 1$, then $D_i = 0$

(3) The values of D_i and R_i specify if V_i:

(a) has a continuous probability density function ($D_i = 1$ or $R_i = 1$), or

(b) has a probability mass ($D_i = R_i = 0$)

$\Rightarrow \{D_i, R_i, V_i\}$ are **not independent** and must be considered together as a triple

• As with two-state model, we define the joint probability function for V_i, D_i and R_i, $f_i(d_i, r_i, v_i)$

• We derive the joint probability function as per two-state version in Lecture08

• Given $d_i = 1$, then $r_i = 0$ and the joint probability function is as per the density for the two-state case:

$$f_i(d_i = 1, r_i = 0, v_i) = S(v_i)\mu_{x_i+v_i} \tag{1}$$

• Given $r_i = 1$, then $d_i = 0$ and the joint probability function is as per the density for the two-state case, but with the force of retirement in place of the force of mortality:

$$f_i(d_i = 0, r_i = 1, v_i) = S(v_i)\nu_{x_i+v_i} \tag{2}$$

• The only other case is $d_i = r_i = 0$, and so there is censoring as the life has neither died nor retired. As with the two-state case, the probability function is a mass corresponding to the survival function:

$$f_i(d_i = 0, r_i = 0, v_i) = S(v_i) \tag{3}$$

• We are dealing here with constant forces of mortality and retirement, so $\mu_x = \mu, \forall x$, and $\nu_x = \nu, \forall \nu$. Also, $\mu^0 = \nu^0 = 1$, so (1), (2) and (3) can be combined and the probability function becomes:

$$f_i(d_i, r_i, v_i) = S(v_i)\mu^{d_i}\nu^{r_i}$$

• The twist is noting that the survival function is that of a combined hazard of leaving the working state, i.e. $\mu + \nu$

• The survival function is therefore $S(v_i) = e^{-(\mu+\nu)t}$, which completes our joint probability function:

$$
\begin{aligned}
f_i(d_i, r_i, v_i) &= e^{-(\mu+\nu)t}\mu^{d_i}\nu^{r_i} \\
&= e^{-\mu t}\mu^{d_i} \times e^{-\nu t}\nu^{r_i}
\end{aligned}
$$

• Note that for this particular model the probability function (and hence the likelihood function) factorises into a product of two functions, each of which resembles the two-state function.

• The likelihood function, L, is the product of the probability functions evaluated at the data points:

$$
\begin{aligned}
L(\mu, \nu; d, r, v) &= \prod_{i=1}^{n} f_i(d_i, r_i, v_i) \\
&= \prod_{i=1}^{n} e^{-\mu v_i}.\mu^{d_i} \times e^{-\nu v_i}.\nu^{r_i} \\
&= e^{-\mu \sum v_i}.\mu^{\sum d_i} \times e^{-\nu \sum v_i}.\nu^{\sum r_i}
\end{aligned}
$$

where:

$\sum v_i$ is the total waiting time,
$\sum d_i$ is the total number of deaths, and
$\sum r_i$ is the total number of retirements.

• Note that the likelihood is a product of Poisson-like likelihoods for μ and ν

• The log-likelihood is therefore:

$$
\ell(\mu, \nu; d, r, v) = -\mu \sum v_i + \log \mu \sum d_i - \nu \sum v_i + \log \nu \sum r_i
$$

• The MLE of μ is $\hat{\mu}$ satisfying $\dfrac{\partial \ell}{\partial \mu} = 0$, i.e. $\hat{\mu} = \dfrac{\sum d_i}{\sum v_i}$

• The MLE of ν is $\hat{\nu}$ satisfying $\dfrac{\partial \ell}{\partial \nu} = 0$, i.e. $\hat{\nu} = \dfrac{\sum r_i}{\sum v_i}$

- Important to be able to write down the likelihoods for a multi-state model

  ⇒ Exam material!
- Some observations for these likelihoods:

(1) There one e term for each non-absoring state,
i.e. for each state which has at least one exit route.

(2) The e term for a non-absorbing state g is a simple survivor-type function based on the total amount of time spent in state g, w_g, say, and the combined forces leading **out** of state g, $\sum_h \mu_{gh}$:

$$e^{-\left(\sum_j \mu_{gh}\right) w_g}$$

where μ_{gh} is the transition intensity from state g to state h.

(3) There is one power term for each transition possible.

(4) The MLE for the transition intensity from state g to state h has form:

$$\frac{\text{Number of transfers from } g \to h}{\text{Total waiting time in state } g}$$