Tutorial 2
Solutions to tutorials are generally available at Iain Currie's homepage at:

http://www.ma.hw.ac.uk/∼iain/teaching/mortality/mortality.html

However, it is worth spelling out a couple of features of the solutions which might not be immediately obvious:

Question 2(ii)
The log-likelihood, ℓ, is given as:

$$\ell = \beta - 2\log(1+e^\beta) - \log(3+4e^\beta)$$

and the first derivative (i.e. score function) is thus:

$$\frac{\partial \ell}{\partial \beta} = 1 - \frac{2e^\beta}{1+e^\beta} - \frac{4e^\beta}{3+4e^\beta}$$

In order to get at the information function, $I(\beta)$, we need the second derivative, $\frac{\partial^2 \ell}{\partial \beta^2}$. But how do we differentiate the above with the inconvenient e^β terms on both numerator and denominator? The trick is to play with the numerators by adding and subtracting constants so we can get terms which mimic the denominators. Thus:

$$\begin{aligned}
\frac{\partial \ell}{\partial \beta} &= 1 - \frac{2e^\beta}{1+e^\beta} - \frac{4e^\beta}{3+4e^\beta} \\
&= 1 - \frac{2+2e^\beta-2}{1+e^\beta} - \frac{3+4e^\beta-3}{3+4e^\beta} \\
&= 1 - \frac{2(1+e^\beta)-2}{1+e^\beta} - \frac{(3+4e^\beta)-3}{3+4e^\beta} \\
&= 1 - \frac{2(1+e^\beta)}{1+e^\beta} + \frac{2}{1+e^\beta} - \frac{3+4e^\beta}{3+4e^\beta} + \frac{3}{3+4e^\beta} \\
&= 1 - 2 + \frac{2}{1+e^\beta} - 1 + \frac{3}{3+4e^\beta} \\
&= -2 + \frac{2}{1+e^\beta} + \frac{3}{3+4e^\beta}
\end{aligned}$$

Without the e^β terms on the numerators, this is now much easier to differentiate:

$$\begin{aligned}
\frac{\partial^2 \ell}{\partial \beta^2} = \frac{\partial}{\partial \beta}\left(\frac{\partial \ell}{\partial \beta}\right) &= \frac{\partial}{\partial \beta}\left(-2 + \frac{2}{1+e^\beta} + \frac{3}{3+4e^\beta}\right) \\
&= -\frac{2e^\beta}{(1+e^\beta)^2} - \frac{12e^\beta}{(3+4e^\beta)^2}
\end{aligned}$$

This approach should work for most Cox log-(partial)likelihoods you are likely to encounter.

Question 2(ii)—continued
The notation is clearer if we write that the hypotheses are $H_0 : \beta_0 = 0$ against $H_1 : \beta_0 \neq 0$.

Question 2(iv)
It isn't clear that ℓ_1 is implicitly the log-likelihood evaluated at $\beta = \hat{\beta}$, rather than $\beta = 1$. After all, ℓ_0 is the log-likelihood evaluated at $\beta = \beta_0 = 0$. For the record, ℓ_0 is the log-likelihood evaluated at $\beta = 0$ and ℓ_1 is the log-likelihood evaluated at $\beta = \hat{\beta}$.

Question 3(ii)
The Taylor expansion of the log-likelihood function, $\ell(\beta)$, is around $\hat{\beta}$ for a good reason. The term $(\beta - \hat{\beta})\ell'(\hat{\beta})$ is zero because $\ell'(\hat{\beta})$ is, by definition, zero. Remember that, by definition, $\hat{\beta}$ is the value at which $\ell'(\beta)$ is zero, i.e. $\hat{\beta}$ is the solution to $\frac{\partial \ell}{\partial \beta} = 0$. It is to take advantage of this result that the Taylor series has been expanded around $\hat{\beta}$, and not any other value.

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